

CONVERGENCE ORDER OF THE GEOMETRIC MEAN ERRORS FOR MARKOV-TYPE MEASURES

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ABSTRACT. We study the quantization problem with respect to the geometric mean error for Markov-type measures μ on a class of fractal sets. Assuming the irreducibility of the corresponding transition matrix P , we determine the exact convergence order of the geometric mean errors of μ . In particular, we show that, the quantization dimension of order zero is independent of the initial probability vector when P is irreducible, while this is not true if P is reducible.

1. INTRODUCTION

In this paper, we study the asymptotic geometric mean errors in the quantization for Markov-type measures on a class of fractal sets. We refer to [5, 7] for mathematical foundations of quantization theory and [8] for its background in engineering technology.

For every $n \geq 1$, we set $\mathcal{D}_n := \{\alpha \subset \mathbb{R}^q : 1 \leq \text{card}(\alpha) \leq n\}$. Let ν be a Borel probability measure on \mathbb{R}^q . The n th quantization error for ν of order r is defined by (see [5, 7]):

$$(1.1) \quad e_{n,r}(\nu) := \begin{cases} \inf_{\alpha \in \mathcal{D}_n} \left(\int d(x, \alpha)^r d\nu(x) \right)^{\frac{1}{r}}, & r > 0, \\ \inf_{\alpha \in \mathcal{D}_n} \exp \int \log d(x, \alpha) d\nu(x), & r = 0. \end{cases}$$

Here $d(\cdot, \cdot)$ is the metric induced by an arbitrary norm on \mathbb{R}^q . For $r > 0$, $e_{n,r}(\nu)$ agrees with the error in the approximation of ν by discrete probability measures supported on at most n points, in the sense of L_r -metrics [5].

By [7, Lemma 3.5], the quantity $e_{n,0}(\nu)$ —also called the n th geometric mean error for ν , equals the limit of $e_{n,r}(\nu)$ as r tends to zero. In this sense, the quantization with respect to the geometric mean error is a limiting case of that in L_r -metrics. As one of the main aims of the quantization problem, we are concerned with the asymptotic properties of the quantization errors, including the upper (lower) quantization coefficient (of order r) and the upper (lower) quantization dimension (of order r).

For $s > 0$, we define the s -dimensional upper and lower quantization coefficient for ν of order $r \in [0, \infty)$ by (cf. [5, 14])

$$\overline{Q}_r^s(\nu) := \limsup_{n \rightarrow \infty} n^{\frac{1}{s}} e_{n,r}(\nu), \quad \underline{Q}_r^s(\nu) := \liminf_{n \rightarrow \infty} n^{\frac{1}{s}} e_{n,r}(\nu).$$

By [5, 14], the upper (lower) quantization dimension $\overline{D}_r(\nu)$ ($\underline{D}_r(\nu)$) of order r , as defined below, is exactly the critical point at which the upper (lower) quantization

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coefficient jumps from zero to infinity:

$$\overline{D}_r(\nu) := \limsup_{n \rightarrow \infty} \frac{\log n}{-\log e_{n,r}(\nu)}, \quad \underline{D}_r(\nu) := \liminf_{n \rightarrow \infty} \frac{\log n}{-\log e_{n,r}(\nu)}.$$

If $\overline{D}_r(\nu) = \underline{D}_r(\nu)$, the common value is denoted by $D_r(\nu)$ and called the quantization dimension for ν of order r .

Compared with the upper (lower) quantization dimension of order r , the upper (lower) quantization coefficient of order r provides us with more accurate information on the asymptotics of the geometric mean errors; accordingly, much more effort is required to examine the finiteness and positivity of the latter.

Remark 1.1. The upper (lower) quantization dimension of ν of order zero is closely connected with the upper (lower) local dimension (cf. [4]) as defined by

$$\underline{\dim}_{\text{loc}} \nu(x) := \liminf_{\epsilon \rightarrow 0} \frac{\log \nu(B_\epsilon(x))}{\log \epsilon}, \quad \overline{\dim}_{\text{loc}} \nu(x) := \limsup_{\epsilon \rightarrow 0} \frac{\log \nu(B_\epsilon(x))}{\log \epsilon}.$$

Here $B_\epsilon(x)$ denotes the closed ball of radius ϵ which is centered at a point $x \in \mathbb{R}^q$. As we showed in [17], if the upper and lower local dimension are both equal to s for ν -a.e. x , then $D_0(\nu)$ exists and equals s .

Next, let us recall a result of Graf and Luschgy. Let $(f_i)_{i=1}^N$ be a family of contractive similitudes on \mathbb{R}^q with contraction ratios $(s_i)_{i=1}^N$. By [10], there exists a unique non-empty compact set K satisfying

$$(1.2) \quad K = f_1(K) \cup f_2(K) \cdots \cup f_N(K).$$

This set is called the self-similar set associated with $(f_i)_{i=1}^N$. Also, By [10], for a probability vector $(q_i)_{i=1}^N$, there exists a unique Borel probability measure ν satisfying $\nu = \sum_{i=1}^N q_i \nu \circ f_i^{-1}$. This measure is called the self-similar measure associated with $(f_i)_{i=1}^N$ and $(q_i)_{i=1}^N$.

We say that $(f_i)_{i=1}^N$ satisfies the open set condition (OSC), if there exists a non-empty bounded open set U such that $f_i(U) \cap f_j(U) = \emptyset$ for all $1 \leq i \neq j \leq N$ and $f_i(U) \subset U$ for all $1 \leq i \leq N$. Let $k_r, r \geq 0$, be given by

$$k_0 := \frac{\sum_{i=1}^N q_i \log q_i}{\sum_{i=1}^N q_i \log s_i}; \quad \sum_{i=1}^N (q_i s_i^r)^{\frac{k_r}{k_r+r}} = 1, \quad r > 0.$$

Assume that $(f_i)_{i=1}^N$ satisfies the OSC. Let ν be the self-similar measure associated with $(f_i)_{i=1}^N$ and $(q_i)_{i=1}^N$. Graf and Luschgy proved [6, 7] that

$$D_r(\nu) = k_r; \quad 0 < \underline{Q}_r^{k_r}(\nu) \leq \overline{Q}_r^{k_r}(\nu) < \infty.$$

In the present paper, we study the finiteness and positivity of the upper and lower quantization coefficient of order zero for Markov-type measures. Some results on the quantization for such measures in L_r -metrics (with $r > 0$) are contained in [12]. Recently, a complete treatment in this direction is given in [11], where the corresponding transition matrix is allowed to be reducible. Next, let us recall some definitions; we refer to [1, 3, 13] for more details.

Let $P = (p_{ij})_{N \times N}$ be a row-stochastic matrix, i.e., $p_{ij} \geq 0, 1 \leq i, j \leq N$; and $\sum_{j=1}^N p_{ij} = 1, 1 \leq i \leq N$. It is easy to see that, 1 is an eigenvalue of P of largest absolute value (cf. [9, Theorem 8.1.22]). When P is irreducible, by the Perron-Frobenius theorem (cf. [9, Theorem 8.4.4]), there exists a unique normalized

positive left (row) eigenvector $v = (v_1, \dots, v_N)$ of P with respect to the eigenvalue 1. We will need the following notations:

$$\begin{aligned} \theta &:= \text{empty word, } G_0 := \{\theta\}, G_1 := \{1, \dots, N\}; \\ G_k &:= \{\sigma \in G_1^k : p_{\sigma_1 \sigma_2} \cdots p_{\sigma_{k-1} \sigma_k} > 0\}, k \geq 2; \\ G^* &:= \bigcup_{k \geq 0} G_k, G_\infty := \{\sigma \in G_1^\mathbb{N} : p_{\sigma_h \sigma_{h+1}} > 0 \text{ for all } h \geq 1\}. \end{aligned}$$

We define $|\sigma| := k$ for $\sigma \in G_k$ and $|\sigma| := \infty$ for $\sigma \in G_1^\mathbb{N}$. For every $\sigma \in G^*$ with $|\sigma| \geq k$ or $\sigma \in G_\infty$, we write $\sigma|_k := (\sigma_1, \dots, \sigma_k)$. For $\sigma \in G^*$ and $\omega \in G^* \cup G_\infty$ with $(\sigma|_{|\sigma|}, \omega_1) \in G_2$, then we set

$$\sigma * \omega = (\sigma_1, \sigma_2, \dots, \sigma_{|\sigma|}, \omega_1, \dots, \omega_{|\omega|}).$$

Let $J_i, i \in G_1$, be non-empty compact subsets of \mathbb{R}^q with $J_i = \overline{\text{int}(J_i)}$ for all $i \in G_1$, where \overline{B} and $\text{int}(B)$ respectively denote the closure and interior in \mathbb{R}^q of a set $B \subset \mathbb{R}^q$. We call these sets cylinders of order one. For each $i \in G_1$, let $J_{ij}, (i, j) \in G_2$, be non-overlapping subsets of J_i such that J_{ij} is geometrically similar to J_j and $|J_{ij}|/|J_j| = c_{ij}$, where $|A|$ denotes the diameter of a set A and $c_{ij} \in (0, 1)$. We call these sets cylinders of order two. Assume that cylinders of order k are determined. Let $J_{\sigma * i_{k+1}}, \sigma * i_{k+1} \in G_{k+1}$, be non-overlapping subsets of J_σ such that $J_{\sigma * i_{k+1}}$ is geometrically similar to $J_{i_{k+1}}$. Hence, by induction, cylinders of order k are determined for all $k \geq 1$. Then, we get a Mauldin-Williams fractal set E (cf. [1, 13]):

$$E := \bigcap_{k \geq 1} \bigcup_{\sigma \in G_k} J_\sigma.$$

The set E need not be a self-similar set, and in general, E does not enjoy the nice invariance property as in (1.2). This will cause much difficulty in the study of the geometric mean error. For this reason, we assume the following separation property for E : there exists some constant $0 < t < 1$ such that for every $\sigma \in G^*$ and j_l with $(\sigma|_{|\sigma|}, j_l) \in G_2, l = 1, 2$,

$$(1.3) \quad d(J_{\sigma * j_1}, J_{\sigma * j_2}) \geq t \max\{|J_{\sigma * j_1}|, |J_{\sigma * j_2}|\}.$$

Let $(q_i)_{i=1}^N$ be an arbitrary probability vector with $q_i > 0$ for all $i \in G_1$. By Kolmogorov consistency theorem, there exists a unique Markov-type measure $\tilde{\mu}$ on G_∞ (cf. [15]) such that, for every $k \geq 1$ and $\sigma = (\sigma_1 \dots \sigma_k) \in G_k$,

$$\tilde{\mu}([\sigma]) = q_{\sigma_1} p_{\sigma_1 \sigma_2} \cdots p_{\sigma_{k-1} \sigma_k},$$

where $[\sigma] := \{\sigma * \omega : \omega \in G_\infty, (\sigma|_{|\sigma|}, \omega_1) \in G_2\}$. With the assumption (1.3), we have the following bijection $g : G_\infty \rightarrow E$:

$$g(\sigma) := \bigcap_{k \geq 1} J_{\sigma|_k}, \quad \sigma \in G_\infty.$$

Let $\mu := \tilde{\mu} \circ g^{-1}$. Then μ is a Markov-type measure on E satisfying

$$(1.4) \quad \mu(J_\sigma) = q_{\sigma_1} p_{\sigma_1 \sigma_2} \cdots p_{\sigma_{k-1} \sigma_k} \text{ for } \sigma = (\sigma_1 \dots \sigma_k) \in G_k, k \geq 1.$$

For each $\sigma \in G^* \setminus \theta$, the way in which μ distributes its measure among the sub-cylinders of J_σ depends on the last entry of σ . This is different from that of the measures as considered in [18]. From now on, we assume

$$(1.5) \quad \text{card}(\{j \in G_1 : (i, j) \in G_2\}) \geq 2 \text{ for all } i \in G_1.$$

Under the assumption (1.5), for each $\sigma \in G^*$, the cylinder J_σ has at least two sub-cylinders of order $|\sigma| + 1$; in addition, we have that $\max_{(i,j) \in G_2} p_{ij} < 1$.

As the main result of the present paper, we will determine the exact convergence order of the geometric mean error for μ . That is,

Theorem 1.2. *Assume that (1.3) and (1.5) are satisfied. Let μ be as given in (1.4). Assume that the transition matrix P is irreducible. Then we have, $D_0(\mu) = s_0$ and $0 < \underline{Q}_0^{s_0}(\mu) \leq \overline{Q}_0^{s_0}(\mu) < \infty$, where*

$$(1.6) \quad s_0 := \frac{\sum_{i=1}^N v_i \sum_{j:(i,j) \in G_2} p_{ij} \log p_{ij}}{\sum_{i=1}^N v_i \sum_{j:(i,j) \in G_2} p_{ij} \log c_{ij}}.$$

and $(v_i)_{i=1}^N$ is the normalized positive left eigenvector of P with respect to 1.

By Theorem 1.2, when the transition matrix P is irreducible, $D_0(\mu)$ is independent of the initial probability vector $(q_i)_{i=1}^N$. In this case, according to [2, 3], $D_0(\mu)$ coincides with the Hausdorff dimension of μ . At the end of the paper, we will give an example to show that, $(q_i)_{i=1}^N$ usually plays a role in the quantization with respect to the geometric mean error for μ when the transition matrix is reducible.

2. A CHARACTERIZATION OF THE GEOMETRIC MEAN ERROR

For every $k \geq 2$ and $\sigma = (\sigma_1, \dots, \sigma_k) \in G_k$, we write

$$\sigma^- := \sigma|_{k-1}; \quad p_\sigma := p_{\sigma_1 \sigma_2} \cdots p_{\sigma_{k-1} \sigma_k}, \quad c_\sigma := c_{\sigma_1 \sigma_2} \cdots c_{\sigma_{k-1} \sigma_k}.$$

If $|\sigma| = 1$, we define $p_\sigma = c_\sigma = 1$ and $\sigma^- = \theta$. If $\sigma, \omega \in G^*$ satisfy $|\sigma| \leq |\omega|$ and $\sigma = \omega|_{|\sigma|}$, then write $\sigma \prec \omega$. Set

$$(2.1) \quad \begin{aligned} \underline{p} &:= \min_{(i,j) \in G_2} p_{ij}, \quad \underline{c} := \min_{(i,j) \in G_2} c_{ij}, \quad \overline{p} := \max_{(i,j) \in G_2} p_{ij}, \quad \overline{c} := \max_{(i,j) \in G_2} c_{ij}; \\ \Lambda_j &:= \{\sigma \in G^* : p_{\sigma^-} \geq \underline{p}^j > p_\sigma\}, \quad \psi_j := \text{card}(\Lambda_j); \\ k_{1j} &:= \min_{\sigma \in \Lambda_j} |\sigma|, \quad k_{2j} := \max_{\sigma \in \Lambda_j} |\sigma|; \\ \underline{P}_0^s(\mu) &:= \liminf_{j \rightarrow \infty} \psi_j^{\frac{1}{s}} e_{\psi_j}(\mu), \quad \overline{P}_0^s(\mu) := \limsup_{j \rightarrow \infty} \psi_j^{\frac{1}{s}} e_{\psi_j}(\mu), \quad s > 0. \end{aligned}$$

Without loss of generality, we assume that $|J_i| = 1$ for all $i \in G_1$. Thus,

$$|J_\sigma| = c_\sigma, \quad \sigma \in G_k, \quad k \geq 1.$$

Lemma 2.1. (i) *There exist constants $A_1, A_2 > 0$ such that*

$$A_1 j \leq k_{1j} \leq k_{2j} \leq A_2 j.$$

(ii) *$\underline{Q}_0^s(\mu) > 0$ iff $\underline{P}_0^s(\mu) > 0$ and $\overline{Q}_0^s(\mu) < \infty$ iff $\overline{P}_0^s(\mu) < \infty$.*

Proof. Let $N_1 := \min\{h \geq 1 : \overline{p}^h < \underline{p}\}$. Let $\sigma^{(l)} \in G_{k_{1j}} \cap \Lambda_j, l = 1, 2$. Then

$$\underline{p}^{k_{1j}} \leq p_{\sigma^{(1)}} < \underline{p}^j, \quad \overline{p}^{k_{2j}-1} \geq p_{\sigma^{(2)}} \geq \underline{p}^{j+1} \geq \overline{p}^{N_1(j+1)}.$$

It follows that $j \leq k_{1j} \leq k_{2j} \leq N_1(j+1)+1 \leq 3N_1 j$, for all $j \geq 1$. Hence (i) follows by setting $A_1 := 1$ and $A_2 := 3N_1$. As in [18], to see (ii), it suffices to show that for some constant $N_2 > 0$ such that $\psi_j \leq \psi_{j+1} \leq N_2 \psi_j$. In fact, the first inequality is clear; to see the second, we note that, for every $\sigma \in \Lambda_j$ and every $\omega \in G_{N_1+1}$ with $(\sigma|_{|\sigma|}, \omega_1) \in G_2$, we have $p_{\sigma * \omega} < \underline{p}^j \overline{p}^{N_1} < \underline{p}^{j+1}$. This implies that $\psi_{j+1} \leq N^{N_1+1} \psi_j$. The lemma follows. \square

2.1. Push-forward and pull-back measures. For each $\sigma \in G^* \setminus \{\theta\}$, we take an arbitrary contracting similitude f_σ on \mathbb{R}^q of contraction ratio c_σ and define $\nu_\sigma := \mu(\cdot | J_\sigma) \circ f_\sigma$. Then, since f_σ is a Borel bijection, ν_σ is a measure supported on $K(\sigma) = f_\sigma^{-1}(J_\sigma)$ satisfying

$$(2.2) \quad \mu(\cdot | J_\sigma) = \nu_\sigma \circ f_\sigma^{-1}, \quad \sigma \in G^* \setminus \{\theta\}.$$

For a finite set $\alpha \subset \mathbb{R}^q$ of cardinality L , by (2.2), we have

$$(2.3) \quad \begin{aligned} \int_{J_\sigma} \log d(x, \alpha) d\mu(x) &= \mu(J_\sigma) \int \log d(x, \alpha) d\nu_\sigma \circ f_\sigma^{-1}(x) \\ &= \mu(J_\sigma) \int \log d(f_\sigma(x), \alpha) d\nu_\sigma(x) \geq \mu(J_\sigma)(\log c_\sigma + \hat{e}_L(\nu_\sigma)). \end{aligned}$$

Lemma 2.2. *There exist constants $A_3, A_4 > 0$, such that*

$$(2.4) \quad \sup_{\sigma \in G^* \setminus \{\theta\}} \sup_{x \in \mathbb{R}^q} \nu_\sigma(B(x, \epsilon)) \leq A_3 \epsilon^{A_4} \quad \text{for all } \epsilon > 0.$$

Proof. Let $\sigma \in G^* \setminus \{\theta\}, x \in K(\sigma)$. By (1.3), there exists a unique word $\tau_x \in G_\infty$ such that $\sigma \prec \tau_x$ and $\bigcap_{k \geq 1} J_{\tau_x|_k} = \{x\}$. For every $\epsilon \in (0, \underline{c})$, we set

$$(2.5) \quad \mathcal{C}(\sigma) := \{\tau \in G^* : \sigma \prec \tau, c_\sigma^{-1} c_{\tau-} \geq \epsilon > c_\sigma^{-1} c_\tau\}.$$

For each $i \in G_1$, there exists some $t_i \in (0, 1)$ such that J_i contains a ball of radius $t_i |J_i| = t_i$ and is contained in a closed ball of radius 1. Set $\delta := \min_{1 \leq i \leq N} t_i$. Then, for each $\tau \in \mathcal{C}(\sigma)$, $f_\sigma^{-1}(J_\tau)$ is contained in a ball of radius ϵ and contains a ball of radius $\delta \underline{c} \epsilon$. By (1.3), $J_\tau, \tau \in \mathcal{C}(\sigma)$ are pairwise disjoint, so are the sets $f_\sigma^{-1}(J_\tau), \tau \in \mathcal{C}(\sigma)$ by the similarity of f_σ . Thus, by [10], there is a constant M which is independent of ϵ such that

$$\text{card}(\{\tau \in \mathcal{C}(\sigma) : B(x, \epsilon) \cap f_\sigma^{-1}(J_\tau) \neq \emptyset\}) \leq M.$$

By (2.5), $\underline{c}^{|\tau| - |\sigma|} < \epsilon$ for $\tau \in \mathcal{C}(\sigma)$, which implies $|\tau| - |\sigma| \geq \log \epsilon / \log \underline{c}$. So,

$$\nu_\sigma(B(x, \epsilon)) \leq M \bar{p}^{\frac{\log \epsilon}{\log \underline{c}}} = M \epsilon^{\frac{\log \bar{p}}{\log \underline{c}}}.$$

Let $A_4 := \frac{\log \bar{p}}{\log \underline{c}}$. Then by [5, Lemma 12.3], there is a constant $A_3 > 0$, independent of σ , such that $\nu_\sigma(B(x, \epsilon)) \leq A_3 \epsilon^{A_4}$ for all $x \in \mathbb{R}^q$. Note that the above arguments holds true for any $\sigma \in G^* \setminus \{\theta\}$. The lemma follows. \square

If the infimum in (1.1) is attained at some α with $1 \leq \text{card}(\alpha) \leq n$, we call α an n -optimal set for ν of order r . The collection of all n -optimal sets for ν of order r is denoted by $C_{n,r}(\nu)$. We simply write $C_n(\nu)$ for $C_{n,0}(\nu)$. Note that $\nu_\sigma, \sigma \in G^*$, are compactly supported. By Lemma 2.4 and [5, Theorem 2.5], we conclude that $C_n(\nu_\sigma)$ is non-empty for every $\sigma \in G^* \setminus \{\theta\}$ and $n \geq 1$. Using similar arguments, one can show that $C_n(\mu)$ is non-empty for every $n \geq 1$.

Lemma 2.3. *(see [7]) Let ν be a Borel probability measure on \mathbb{R}^q with compact support K . Let $\hat{e}_n(\nu) := \log e_{n,0}(\nu)$. Assume that for some constants $d_1, d_2 > 0$ we have, $\sup_{x \in \mathbb{R}^q} \nu(B(x, \epsilon)) \leq d_1 \epsilon^{d_2}$. Then, we have*

$$\hat{e}_n(\nu) - \hat{e}_{n+1}(\nu) \leq (n+1)^{-1} \log(3|K|) + d_1^{1/q} q d_2^{-1} (n+1)^{-1/p}, \quad n \geq 1.$$

where p, q are real numbers satisfying $p, q > 1, p^{-1} + q^{-1} = 1$.

Let $\underline{q} := \min_{i \in G_1} q_i$ and $\bar{q} := \max_{i \in G_1} q_i$. As a consequence of (2.4) and Lemma 2.3, for given integers $k_1, k_2, k_3 \geq 1$, there exists an integer A_5 such that, for all $n \geq A_5$, we have

$$(2.6) \quad \sup_{\sigma \in G^*} (\hat{e}_{n-k_1-k_3}(\nu_\sigma) - \hat{e}_{n+k_2}(\nu_\sigma)) < \underline{q}\bar{q}^{-1}\underline{p} \log 2.$$

Remark 2.4. Using (2.4) and the proof of Theorem 3.4 of [7], it is convenient to see, for every $k \geq 1$, there is a $B_k \in \mathbb{R}$ such that $\inf_{\sigma \in G^*} \hat{e}_k(\nu_\sigma) \geq B_k$.

2.2. An estimate of the geometric mean error. For $\epsilon > 0$, let $(A)_\epsilon$ denote the closed ϵ -neighborhood in \mathbb{R}^q of a set $A \subset \mathbb{R}^q$. Let t be the same as in (1.3). For a finite subset α of \mathbb{R}^q and $\sigma \in G^*$, we write $\alpha_\sigma := \alpha \cap (J_\sigma)_{4^{-1}tc_\sigma}$ and

$$L_\sigma := \text{card}(\alpha_\sigma), \quad I_\sigma(\alpha) := \int_{J_\sigma} \log d(x, \alpha) d\mu(x).$$

By (1.4), we have, $\mu(J_\sigma) = q_{\sigma_1} p_\sigma$ for every $\sigma \in G^* \setminus \{\theta\}$. We set

$$J_\theta := E; \quad m_\sigma := q_{\sigma_1} p_\sigma, \quad \sigma \in G^* \setminus \{\theta\}.$$

Lemma 2.5. *There exists a constant L_1 which is independent of j such that*

$$(2.7) \quad \sup_{\alpha \in C_{\psi_j}(\mu)} \max_{\sigma \in \Lambda_j} L_\sigma \leq L_1.$$

Proof. Since all $\nu_\sigma, \sigma \in G^*$, share the properties in (2.6) and (2.4), it suffices to follow the induction in [16, Proposition 3.4] by using (2.2). \square

Remark 2.6. For the reader's convenience, let us explain the main idea of the induction in [16] by contradiction: suppose that (2.7) does not hold; we could choose a set β with $\text{card}(\beta) \leq \text{card}(\alpha)$ which is "better" than α .

Let H_1 be the smallest integer such that $(J_\sigma)_{2^{-1}tc_\sigma}$ can be covered by H_1 closed balls of radii $8^{-1}tc_\sigma$ which are centered in $(J_\sigma)_{2^{-1}tc_\sigma}$, we denote by $\gamma_1(\sigma)$ the centers of such H_1 closed balls. Let H_2 be the smallest integer such that $(J_\sigma)_{4^{-1}tc_\sigma}$ can be covered by H_2 closed balls of radii $8^{-1}tc_\sigma$. Let us denote by $\gamma_2(\sigma)$ the centers of such H_2 closed balls. Then by (2.2), we have

$$(2.8) \quad I_\sigma(\alpha) \geq I_\sigma(\alpha \cup \gamma_2(\sigma)) \geq m_\sigma(\log c_\sigma + \hat{e}_{L_\sigma+H_2}(\nu_\sigma)).$$

Let H_3 be the smallest integer such that J_τ can be covered by H_3 closed balls of radii $8^{-1}tc_\tau$ which are centered in J_τ and we denote by $\gamma_3(\tau)$ the centers of such H_3 closed balls. Let L_0 be the smallest integer such that (2.4) holds with $k_i = H_i, i = 1, 2, 3$. Set $L_1 := L_0 + H_1 + H_3$.

Suppose that $L_\sigma = \text{card}(\alpha_\sigma) > L_1$. By (1.3), $\alpha_\sigma \cap \alpha_\omega = \emptyset$ for distinct words $\sigma, \omega \in \Lambda_j$. So, there is a $\tau \in \Lambda_j$ with $L_\tau = 0$. Let $\gamma_4(\sigma) \in C_{L_\sigma-H_1-H_3}(\nu_\sigma)$. Set

$$\beta := (\alpha \setminus \alpha_\sigma) \cup \gamma_1(\sigma) \cup \gamma_3(\tau) \cup \gamma_4(\sigma).$$

Then $\text{card}(\alpha) \geq \text{card}(\beta)$. The set $\gamma_1(\sigma)$ ensures that $J_\omega, \omega \in \Lambda_j \setminus \{\sigma, \tau\}$, are not affected unfavorably while we try to adjust the "optimal points" between α_σ and α_τ . In fact, by triangle inequality, we have, $d(x, \alpha_\sigma) \geq d(x, \gamma_1(\sigma))$ for $x \in J_\omega, \omega \in \Lambda_j \setminus \{\sigma, \tau\}$. It follows that

$$(2.9) \quad I_\omega(\alpha) > I_\omega(\beta) \quad \text{for all } \omega \in \Lambda_j \setminus \{\sigma, \tau\}.$$

Thus, it suffices to estimate the following differences separately:

$$\Delta_1 := I_\sigma(\beta) - I_\sigma(\alpha); \quad \Delta_2 := I_\tau(\alpha) - I_\tau(\beta).$$

Using (2.2), (2.6) and (2.8), it is easy to show that $\Delta_1 < \Delta_2$. This, together with (2.9), implies that $I_\theta(\alpha) > I_\theta(\beta)$, contradicting the optimality of α .

Lemma 2.7. *There exists a constant C_0 such that for all large $j \in \mathbb{N}$,*

$$\sum_{\tau \in \Lambda_j} m_\tau \log c_\tau + C_0 \leq \hat{e}_{\psi_j}(\mu) \leq \sum_{\tau \in \Lambda_j} m_\tau \log c_\tau.$$

Proof. Let $\alpha \in C_{\psi_j}(\mu)$ and let $\gamma_3(\tau)$ be as defined in Remark 2.6. By (2.7), $\text{card}(\alpha_\tau \cup \gamma_3(\tau)) \leq L_1 + H_3$ for every $\tau \in \Lambda_j$. One can see that,

$$d(x, \alpha) \geq d(x, \alpha_\tau \cup \gamma_3(\tau)) \text{ for all } x \in J_\tau.$$

Set $C_0 := B_{L_1+H_3}$. By (2.3) and Remark 2.4,

$$\hat{e}_{\psi_j}(\mu) \geq \sum_{\tau \in \Lambda_j} I_\tau(\alpha_\tau \cup \gamma_3(\tau)) \geq \sum_{\tau \in \Lambda_j} m_\tau \log c_\tau + C_0.$$

For each $\tau \in \Lambda_j$, let b_τ be an arbitrary point in J_τ and set $\gamma := \{b_\tau\}_{\tau \in \Lambda_j}$. Then we have, $\hat{e}_{\psi_j}(\mu) \leq I_\theta(\gamma) \leq \sum_{\tau \in \Lambda_j} m_\tau \log c_\tau$. The lemma follows. \square

For $i \in G_1$, let μ_i denote the conditional probability measure $\mu(\cdot | J_i)$, namely, for every Borel set $B \subset \mathbb{R}^q$, $\mu_i(B) = \mu(B \cap J_i) / \mu(J_i)$. We define

$$G_k(i) := \{\sigma \in G_k : \sigma_1 = i\}, \quad k \geq 1; \quad G^*(i) := \bigcup_{k \geq 1} G_k(i);$$

$$\Lambda_k(i) := \{\sigma \in G^*(i) : p_{\sigma^-} \geq \underline{p}^k > p_\sigma\}, \quad \psi_k(i) := \text{card}(\Lambda_k(i)), \quad k \geq 1.$$

For $\sigma \in G^*(i)$, we have, $\mu_i(J_\sigma) = p_\sigma$. As we did for μ , one can show that, there exists a constant $C_0(i)$ such that for all large $k \in \mathbb{N}$,

$$(2.10) \quad \sum_{\sigma \in \Lambda_k(i)} p_\sigma \log c_\sigma + C_0(i) \leq \hat{e}_{\psi_k(i)}(\mu) \leq \sum_{\sigma \in \Lambda_k(i)} p_\sigma \log c_\sigma.$$

3. PROOF OF THEOREM 1.2

For the proof of Theorem 1.2, we need to establish several lemmas. For $k, n \geq 1$, and $\sigma \in G_k$, let $\Gamma(\sigma, n) := \{\omega \in G_{k+n} : \sigma \prec \omega\}$; we define

$$\xi(i, n) := \sum_{\tau \in \Gamma(i, n)} p_\tau \log p_\tau; \quad \lambda(i, n) := \sum_{\tau \in \Gamma(i, n)} p_\tau \log c_\tau, \quad i \in G_1;$$

$$\Delta_n(i, j) := |\xi(i, n) - \xi(j, n)|, \quad \tilde{\Delta}_n(i, j) := |\lambda(i, n) - \lambda(j, n)|, \quad i, j \in G_1.$$

For $1 \leq i, j \leq N$ with $(i, j) \notin G_2$, we have, $p_{ij} = c_{ij} = 0$. In the following, we take the convention that $0 \cdot \log 0 := 0$, so that we may take the sums from $i = 1$ to N , instead of considering words in G^* . We always denote by $v = (v_i)_{i=1}^N$ the normalized positive left (row) eigenvector of P with respect to the Perron-Frobenius eigenvalue 1, when P is irreducible. Let u_0 and l_0 denote the numerator and denominator in the definition of s_0 (see (1.6)).

To study the asymptotics of the geometric mean errors, we will naturally need the following estimate which reflects some hereditary information of μ . One may see [18, (2.11)] for a comparison.

Lemma 3.1. *Assume that P is irreducible. There exists a $C_1 > 0$ such that*

$$(3.1) \quad \sup_{n \geq 1} \max_{i, j \in G_1} \max \{\Delta_n(i, j), \tilde{\Delta}_n(i, j)\} \leq C_1.$$

Proof. For $h \geq 1$ and $l, p \in G_1$, let $b_{lp}^{(h)}$ denote the (l, p) -entry of P^h . For $h \geq 3$ and $l \in G_1$, we have, $\sum_{\tau \in G_{h-2}(i)} p_{\tau * l} = b_{il}^{(h-2)}$ (cf. [13, (30)]). In addition, we have, $G_h(i) = \Gamma(i, h-1)$. One can see

$$\begin{aligned} \xi(i, h-1) &= \sum_{\omega \in G_{h-1}(i)} \sum_{j=1}^N p_{\omega} p_{\omega_{h-1}j} \log p_{\omega} + \sum_{\tau \in G_{h-2}(i)} \sum_{l=1}^N \sum_{j=1}^N p_{\tau * l} p_{lj} \log p_{lj} \\ &= \xi(i, h-2) + \sum_{l=1}^N \sum_{j=1}^N b_{il}^{(h-2)} p_{lj} \log p_{lj}. \end{aligned}$$

Write $d_h(i) := \sum_{l=1}^N \sum_{j=1}^N b_{il}^{(h-2)} p_{lj} \log p_{lj}$, $h \geq 3$. By induction, we have

$$\xi(i, k-1) = \xi(i, 1) + \sum_{h=3}^k d_h(i), \quad k \geq 3.$$

Set $w_{k,i} := \xi(i, 1) + (k-2)u_0$. Then we have

$$(3.2) \quad \xi(i, k-1) = w_{k,i} + \sum_{l=1}^N \sum_{j=1}^N p_{lj} \log p_{lj} \sum_{h=3}^k (b_{il}^{(h-2)} - v_l).$$

Similarly, set $z_{k,i} := \lambda(i, 1) + (k-2)l_0$. Then, for $k \geq 3$, we have

$$(3.3) \quad \lambda(i, k-1) = z_{k,i} + \sum_{l=1}^N \sum_{j=1}^N p_{lj} \log c_{lj} \sum_{h=3}^k (b_{il}^{(h-2)} - v_l).$$

Let $u = (\chi_i)_{i=1}^N$ be the column vector with $\chi_i = 1$ for all $1 \leq i \leq N$. Then u is a right eigenvector of P with respect to 1 and $\sum_{i=1}^N \chi_i v_i = 1$. We have

$$L := uv =: (l_{ij})_{N \times N}, \quad l_{ij} = v_j, \quad 1 \leq i, j \leq N.$$

Applying [9, Theorem 8.6.1] with the above matrix L , there exists a constant $C(P)$ such that for all $k \geq 3$,

$$\frac{1}{k-2} \left| \sum_{h=3}^k (b_{pl}^{(h-2)} - v_l) \right| = \left| \frac{1}{k-2} \sum_{h=3}^k b_{pl}^{(h-2)} - v_l \right| < \frac{C(P)}{k-2}, \quad p, l \in G_1.$$

This, together with (3.2), yields

$$(3.4) \quad \frac{1}{k-2} |\xi(i, k-1) - w_{k,i}| \leq \frac{C(P)}{k-2} \sum_{l=1}^N \sum_{j=1}^N |p_{lj} \log p_{lj}| =: \frac{\delta_0}{k-2}; \quad k \geq 3.$$

Hence, $|\xi(i, k-1) - w_{k,i}| \leq \delta_0$ for all $k \geq 3$. Note that, the above argument is true for all $i \in G_1$. Set $\delta_1 := \max_{i,j \in G_1} |\xi(i, 1) - \xi(j, 1)|$. Then for $n = k-1$,

$$\Delta_n(i, j) \leq |\xi(i, n) - w_{k,i}| + |w_{k,i} - w_{k,j}| + |\xi(j, n) - w_{k,j}| \leq 2\delta_0 + \delta_1 =: \delta_2.$$

Analogously, for some constant $\delta_3 > 0$, we have, $\tilde{\Delta}_n(i, j) \leq \delta_3$ for $i, j \in G_1$ and $h \geq 1$. Thus, the lemma follows by setting $C_1 := \max\{\delta_2, \delta_3\}$. \square

The following two number sequences $(t_k)_{k=1}^\infty$ and $(s_k)_{k=1}^\infty$ are closely connected with the asymptotic geometric mean errors:

$$(3.5) \quad t_k := \frac{\sum_{\sigma \in \Lambda_k} m_{\sigma} \log m_{\sigma}}{\sum_{\sigma \in \Lambda_k} m_{\sigma} \log c_{\sigma}}; \quad s_k := \frac{\sum_{\sigma \in G_k} m_{\sigma} \log m_{\sigma}}{\sum_{\sigma \in G_k} m_{\sigma} \log c_{\sigma}}, \quad k \geq 1.$$

Let u_k, l_k denote the numerator and denominator in the definition of s_k . Then

$$(3.6) \quad u_1 = \sum_{i=1}^N q_i \log q_i, \quad u_2 = \sum_{i=1}^N \sum_{j=1}^N q_i p_{ij} \log(q_i p_{ij}) = u_1 + \sum_{i=1}^N q_i \xi(i, 1).$$

Lemma 3.2. *Assume that P is irreducible. There exists a constant C_2 such that for all large $k \in \mathbb{N}$, we have $|s_k - s_0| \leq C_2 k^{-1}$.*

Proof. Let $w_{k,i}, z_{k,i}$ be as defined in the proof of Lemma 3.1. We write

$$x_k := \sum_{i=1}^N q_i (\xi(i, k-1) - w_{k,i}), \quad y_k := \sum_{i=1}^N q_i (\lambda(i, k-1) - z_{k,i}), \quad k \geq 3.$$

For $k \geq 3$, by (3.2), (3.6) and the definitions of u_k and $\xi(i, k-1)$, we deduce

$$\begin{aligned} u_k &= \sum_{i=1}^N \sum_{\omega \in G_k(i)} q_i p_\omega \log q_i + \sum_{i=1}^N \sum_{\omega \in G_k(i)} q_i p_\omega \log p_\omega \\ &= u_1 + \sum_{i=1}^N q_i \xi(i, k-1) \\ &= u_1 + \sum_{i=1}^N q_i (\xi(i, 1) + (k-2)u_0 + \xi(i, k-1) - w_{k,i}) \\ (3.7) \quad &= u_2 + (k-2)u_0 + x_k. \end{aligned}$$

Note that $l_1 = 0$. By replacing $\log p_{ij}$ in (3.7) with $\log c_{ij}$, we have

$$(3.8) \quad l_k = l_2 + (k-2)l_0 + y_k.$$

By (3.4), we have, $|\xi(i, k-1) - w_{k,i}| \leq \delta_0$ for all $i \in G_1$. Thus,

$$(3.9) \quad |x_k| \leq \sum_{i=1}^N q_i |\xi(i, k-1) - w_{k,i}| \leq \delta_0 \quad \text{and} \quad |y_k| \leq \delta_3.$$

Set $s_2 := u_2 + l_2$ and $A_6 := |s_2(l_0 + u_0)|$. by (3.7)-(3.9), for large k , we deduce

$$\begin{aligned} |s_k - s_0| &= \left| \frac{u_2 + (k-2)u_0 + x_k}{l_2 + (k-2)l_0 + y_k} - \frac{u_0}{l_0} \right| \\ &\leq \frac{A_6 + |x_k l_0 - y_k u_0|}{|l_0(l_2 + (k-2)l_0 + y_k)|} \\ &\leq \frac{2A_6 + 2(\delta_0 + \delta_3)(|u_0| + |l_0|)}{(k-2)l_0^2} \\ &\leq \frac{4A_6 + 4(\delta_0 + \delta_3)(|u_0| + |l_0|)}{kl_0^2} =: C_2 k^{-1}. \end{aligned}$$

This completes the proof of the lemma. \square

Remark 3.3. If $(q_i)_{i=1}^N$ agrees with v , then $\sum_{i=1}^N q_i b_{il}^{(h-2)} = v_l$ since $vP^k = v$ for all $k \geq 1$. Hence, $x_k = 0$ and (3.7) becomes: $u_k = u_1 + (k-2)u_0$. This was calculated in [15, Theorem 4.27].

Next we establish a connection between $(t_j)_{j=1}^\infty$ and $(s_k)_{k=1}^\infty$. We have

Lemma 3.4. *Assume that P is irreducible. There exist a constant C_3 and two integers $k_j^{(i)} \in [k_{1j}, k_{2j}]$, $i = 1, 2$ such that*

$$s_{k_j^{(1)}} - C_3 j^{-1} \leq t_j \leq s_{k_j^{(2)}} + C_3 j^{-1}.$$

Proof. For $k \geq 1$ and $\sigma \in G_k$, we have

$$\begin{aligned} \sum_{\tau \in \Gamma(\sigma, k_{2j} - |\sigma|)} m_\tau \log m_\tau &= \sum_{\tau \in \Gamma(\sigma, k_{2j} - |\sigma|)} m_\tau \left(\log m_\sigma + \log \frac{m_\tau}{m_\sigma} \right) \\ &= m_\sigma \log m_\sigma + m_\sigma \sum_{\tau \in \Gamma(\sigma, k_{2j} - |\sigma|)} \frac{m_\tau}{m_\sigma} \log \frac{m_\tau}{m_\sigma} \\ (3.10) \quad &= m_\sigma \log m_\sigma + m_\sigma \xi(\sigma_{|\sigma|}, k_{2j} - |\sigma|). \end{aligned}$$

By (3.10) and Lemma 3.1, for every $k \in [k_{1j}, k_{2j}]$ and $\omega \in G_k$, we have

$$\begin{aligned} u_{k_{2j}} - u_k &= \sum_{\sigma \in G_k} \sum_{\tau \in \Gamma(\sigma, k_{2j} - k)} m_\tau \log m_\tau - u_k = \sum_{\sigma \in G_k} m_\sigma \xi(\sigma_k, k_{2j} - k) \\ (3.11) \quad &\leq \sum_{\sigma \in G_k} m_\sigma (\xi(\omega_k, k_{2j} - k) + C_1) = \xi(\omega_k, k_{2j} - k) + C_1. \end{aligned}$$

Similarly, we have that $u_{k_{2j}} - u_k \geq \xi(\omega_k, k_{2j} - k) - C_1$. We define

$$\zeta(\sigma) := m_\sigma (\log m_\sigma - u_{|\sigma|}), \quad \sigma \in G^* \setminus \{\theta\}.$$

Then, by (3.10) and (3.11), we deduce

$$\begin{aligned} \sum_{\tau \in \Gamma(\sigma, k_{2j} - |\sigma|)} \zeta(\tau) &= \sum_{\tau \in \Gamma(\sigma, k_{2j} - |\sigma|)} m_\tau (\log m_\tau - u_{k_{2j}}) \\ &= m_\sigma \log m_\sigma + m_\sigma \xi(\sigma_{|\sigma|}, k_{2j} - |\sigma|) - m_\sigma u_{k_{2j}} \\ \begin{cases} \leq m_\sigma \log m_\sigma + m_\sigma (u_{k_{2j}} - u_{|\sigma|} + C_1) - m_\sigma u_{k_{2j}} = \zeta(\sigma) + m_\sigma C_1. \\ \geq m_\sigma \log m_\sigma + m_\sigma (u_{k_{2j}} - u_{|\sigma|} - C_1) - m_\sigma u_{k_{2j}} = \zeta(\sigma) - m_\sigma C_1 \end{cases} \end{aligned}$$

This is equivalent to

$$\sum_{\tau \in \Gamma(\sigma, k_{2j} - |\sigma|)} \zeta(\tau) - m_\sigma C_1 \leq \zeta(\sigma) \leq \sum_{\tau \in \Gamma(\sigma, k_{2j} - |\sigma|)} \zeta(\tau) + m_\sigma C_1.$$

Note that $\sum_{\tau \in G_{k_{2j}}} \zeta(\tau) = 0$. We further deduce

$$\begin{aligned} \sum_{\sigma \in \Lambda_j} m_\sigma (\log m_\sigma - u_{|\sigma|}) &= \sum_{\sigma \in \Lambda_j} \zeta(\sigma) \\ \begin{cases} \leq \sum_{\sigma \in \Lambda_j} \left(\sum_{\tau \in \Gamma(\sigma, k_{2j} - |\sigma|)} \zeta(\tau) + m_\sigma C_1 \right) = C_1. \\ \geq \sum_{\sigma \in \Lambda_j} \left(\sum_{\tau \in \Gamma(\sigma, k_{2j} - |\sigma|)} \zeta(\tau) - m_\sigma C_1 \right) = -C_1. \end{cases} \end{aligned}$$

As an immediate consequence, we have

$$(3.12) \quad \sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} - C_1 \leq \sum_{\sigma \in \Lambda_j} m_\sigma \log m_\sigma \leq \sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} + C_1.$$

Similarly, one can show that

$$(3.13) \quad \sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} - C_1 \leq \sum_{\sigma \in \Lambda_j} m_\sigma \log c_\sigma \leq \sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} + C_1.$$

Thus, for large j , by (3.12) and (3.13), we have

$$(3.14) \quad \frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} + C_1}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} - C_1} \leq \frac{\sum_{\sigma \in \Lambda_j} m_\sigma \log m_\sigma}{\sum_{\sigma \in \Lambda_j} m_\sigma \log c_\sigma} \leq \frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} - C_1}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} + C_1}.$$

Let $A_7 := 2^{-1}|l_0|$. Note that $l_k, k \geq 2$, are all negative. By (3.8) and (3.9),

$$(3.15) \quad |l_k| \geq (k-2)|l_0| - \delta_0 \geq 2^{-1}k|l_0| = A_7 k, \quad k \geq 4 + \delta_0|l_0|^{-1}.$$

This, together with the definition of k_{1j} , implies

$$(3.16) \quad \left| \sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} - C_1 \right| \geq \sum_{\sigma \in \Lambda_j} m_\sigma |l_{|\sigma|}| \geq A_7 k_{1j}.$$

By Lemma 3.2, we have, $s_k \leq 2s_0$ for all large k . Hence,

$$\frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} \leq \max_{k_{1j} \leq k \leq k_{2j}} s_k \leq 2s_0.$$

Combining this with (3.16), we have

$$(3.17) \quad \begin{aligned} & \left| \frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} + C_1}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} - C_1} - \frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} \right| \\ &= \frac{C_1(|\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}| + |\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}|)}{|(\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} - C_1) \sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}|} \leq \frac{C_1(1+2s_0)}{A_7 k_{1j}}. \end{aligned}$$

For large j , we have, $C_1 \leq 2^{-1}|\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}|$. Hence, we similarly get

$$(3.18) \quad \left| \frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|} - C_1}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|} + C_1} - \frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} \right| \leq \frac{2C_1(1+2s_0)}{A_7 k_{1j}}.$$

Set $A_8 := 2C_1(1+2s_0)A_7^{-1}$. By (3.14), (3.17) and (3.18), we deduce

$$\frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} - \frac{A_8}{k_{1j}} \leq \frac{\sum_{\sigma \in \Lambda_j} m_\sigma \log m_\sigma}{\sum_{\sigma \in \Lambda_j} m_\sigma \log c_\sigma} \leq \frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} + \frac{A_8}{k_{1j}}.$$

Now one can see that, there exist some $k_j^{(i)} \in [k_{1j}, k_{2j}]$, $i = 1, 2$, such that

$$s_{k_j^{(1)}} = \min_{k_{1j} \leq k \leq k_{2j}} \frac{u_k}{l_k} \leq \frac{\sum_{\sigma \in \Lambda_j} m_\sigma u_{|\sigma|}}{\sum_{\sigma \in \Lambda_j} m_\sigma l_{|\sigma|}} \leq \max_{k_{1j} \leq k \leq k_{2j}} \frac{u_k}{l_k} = s_{k_j^{(2)}}.$$

Thus, in view of (2.1), the lemma follows by setting $C_3 := A_8/A_1$. \square

With the above analysis, we obtain the convergence order of $(t_j)_{j=1}^\infty$:

Lemma 3.5. *Assume that P is irreducible. There exists a constant C_4 such that $|t_j - s_0| < C_4 j^{-1}$ for all large j .*

Proof. By Lemmas 3.2, 3.4 and Lemma 2.1, we have

$$t_j - s_0 \begin{cases} \leq s_{k_j^{(2)}} - s_0 + C_3 j^{-1} \leq (C_2 A_1^{-1} + C_3) j^{-1} \\ \geq s_{k_j^{(1)}} - s_0 - C_3 j^{-1} \geq -(C_2 A_1^{-1} + C_3) j^{-1} \end{cases}.$$

Hence, the lemma follows by setting $C_4 := C_2 A_1^{-1} + C_3$. \square

Now we are able to give the proof of Theorem 1.2. For a Borel probability measure ν on \mathbb{R}^q and every $n \geq 1$, we write

$$Q_n(\nu, a) := a^{-1} \log n + \hat{e}_n(\nu), \quad a > 0.$$

Proof of Theorem 1.2 By (2.1), we easily see

$$(3.19) \quad \underline{p}^{j+1} \leq p_\sigma < \underline{p}^j; \quad \bar{q}^{-1} \underline{p}^{-j} \leq \psi_j \leq \underline{q}^{-1} \underline{p}^{-(j+1)}.$$

Since $t_j \rightarrow s_0$, we have, $2^{-1}s_0 \leq t_j \leq 2s_0$ for all large j . By (3.19),

$$\begin{aligned} \sum_{\sigma \in \Lambda_j} m_\sigma \log c_\sigma &= t_j^{-1} \sum_{\sigma \in \Lambda_j} m_\sigma \log m_\sigma = t_j^{-1} \sum_{\sigma \in \Lambda_j} m_\sigma \log(q_{\sigma_1} p_\sigma) \\ &= t_j^{-1} \sum_{\sigma \in \Lambda_j} m_\sigma \log q_{\sigma_1} + t_j^{-1} \sum_{\sigma \in \Lambda_j} m_\sigma \log p_\sigma \begin{cases} \leq (2s_0)^{-1} \log \bar{q} + t_j^{-1} \log \underline{p}^j \\ \geq 2s_0^{-1} \log \underline{q} + t_j^{-1} \log \underline{p}^{j+1} \end{cases}. \end{aligned}$$

This, together with Lemma 2.7, yields

$$2s_0^{-1} \log \underline{q} + t_j^{-1} \log \underline{p}^{j+1} + C_0 \leq \hat{e}_{\psi_j}(\mu) \leq t_j^{-1} \log \underline{p}^j + (2s_0)^{-1} \log \bar{q}.$$

Thus, by Lemma 3.4, (3.19) and the fact that $2^{-1}s_0 \leq t_j \leq 2s_0$, we deduce

$$\begin{aligned} Q_{\psi_j}(\mu, s_0) &= s_0^{-1} \log \psi_j + \hat{e}_{\psi_j}(\mu) \\ &\begin{cases} \leq (s_0^{-1} - t_j^{-1}) \log \underline{p}^{-j} - s_0^{-1} \log(\underline{q}\underline{p}) + (2s_0)^{-1} \log \bar{q} \\ \geq (s_0^{-1} - t_j^{-1}) \log \underline{p}^{-j} + 2s_0^{-1} \log \underline{q} + 2s_0^{-1} \log \underline{p} + C_0 \end{cases}. \end{aligned}$$

Finally, using Lemma 3.5, we obtain

$$Q_{\psi_j}(\mu, s_0) \begin{cases} \leq 2C_4 s_0^{-2} \log \underline{p}^{-1} - s_0^{-1} \log(\underline{q}\underline{p}) + (2s_0)^{-1} \log \bar{q} \\ \geq -2C_4 s_0^{-2} \log \underline{p}^{-1} + 2s_0^{-1} \log \underline{q} + 2s_0^{-1} \log \underline{p} + C_0 \end{cases}.$$

Hence, $0 < \underline{P}_0^{s_0}(\mu) \leq \overline{P}_0^{s_0}(\mu) < \infty$. By Lemma 2.1, the theorem follows.

In the following, we study the asymptotic geometric mean error for the conditional probability measures μ_i . For every $i \in G_1$, we write

$$t_k(i) := \frac{\sum_{\sigma \in \Lambda_k(i)} p_\sigma \log p_\sigma}{\sum_{\sigma \in \Lambda_k(i)} p_\sigma \log c_\sigma}; \quad s_k(i) := \frac{\sum_{\sigma \in G_k(i)} p_\sigma \log p_\sigma}{\sum_{\sigma \in G_k(i)} p_\sigma \log c_\sigma}, \quad k \geq 2.$$

Let us denote by $u_k(i), l_k(i)$ the numerator and denominator in the definition of $s_k(i)$. We have the following estimate for the convergence order of $(t_j(i))_{j=1}^\infty$.

Lemma 3.6. *Assume that P is irreducible. There exists a constant C_5 such that $|t_j(i) - s_0| < C_5 j^{-1}$ for every $i \in G_1$ and all large j .*

Proof. Fix an arbitrary $i \in G_1$. By Lemma 3.1, for every pair $l, h \in G_1$,

$$u_k(h) - C_1 \leq u_k(l) = \xi(l, k-1) \leq \xi(h, k-1) + C_1 = u_k(h) + C_1.$$

Thus, for the above i and $k \geq 2$, we have

$$Nu_k(i) - NC_1 \leq u_k = \sum_{j=1}^N u_k(j) \leq Nu_k(i) + NC_1.$$

Set $a_k := u_k(i) - N^{-1}u_k$ and $b_k := l_k(i) - N^{-1}l_k$. Then $|a_k|, |b_k| \leq C_1$. Note that $|l_k| \rightarrow \infty$ as $k \rightarrow \infty$. Hence, for large k , we have,

$$|N^{-1}l_k + b_k| \geq 2^{-1}|N^{-1}l_k| \quad \text{and} \quad s_k \leq 2s_0.$$

Using these facts and (3.15), we deduce

$$\begin{aligned}
 |s_k(i) - s_k| &= \left| \frac{u_k(i)}{l_k(i)} - \frac{u_k}{l_k} \right| = \left| \frac{N^{-1}u_k + a_k}{N^{-1}l_k + b_k} - \frac{u_k}{l_k} \right| \\
 &\leq \frac{C_1(|u_k| + |l_k|)}{|l_k(N^{-1}l_k + b_k)|} \leq \frac{C_1}{|N^{-1}l_k + b_k|} + \frac{C_1|u_k|}{|l_k(N^{-1}l_k + b_k)|} \\
 (3.20) \quad &\leq \frac{2N_1C_1}{|l_k|} + \frac{2N_1C_1s_k}{|l_k|} \leq \frac{2N_1C_1(1+2s_0)}{kA_7} =: A_9k^{-1}.
 \end{aligned}$$

Let $k_{1j}(i) := \min_{\sigma \in \Lambda_j(i)} |\sigma|$ and $k_{2j}(i) := \max_{\sigma \in \Lambda_j(i)} |\sigma|$. By Lemma 2.1,

$$A_1j \leq k_{1j} \leq k_{1j}(i) \leq k_{2j}(i) \leq k_{2j} \leq A_2j.$$

Along the line in the proof of Lemma 3.4, one can show, there exist a constant $C_3(i)$ and two integers $k_j^{(h)} \in [k_{1j}(i), k_{2j}(i)]$, $h = 1, 2$, such that

$$s_{k_j^{(1)}}(i) - C_3(i)j^{-1} \leq t_j(i) \leq s_{k_j^{(2)}}(i) + C_3(i)j^{-1}.$$

Combining this and (3.20), we have

$$\begin{aligned}
 t_j(i) - s_0 &\leq s_{k_j^{(2)}}(i) + C_3(i)j^{-1} - s_0 \leq A_9k_j^{(2)}(i)^{-1} + C_3(i)j^{-1} \\
 &\leq A_9A_1^{-1}j^{-1} + C_3(i)j^{-1} = (A_9A_1^{-1} + C_3(i))j^{-1}.
 \end{aligned}$$

Similarly, one can show that $t_j(i) - s_0 \geq -(A_9A_1^{-1} + C_3(i))j^{-1}$. Thus, the lemma follows by setting $C_5 := A_9A_1^{-1} + \max_{i \in G_1} C_3(i)$. \square

Proposition 3.7. *Assume that P is irreducible. Then, we have*

$$0 < \underline{Q}_0^{s_0}(\mu_i) \leq \overline{Q}_0^{s_0}(\mu_i) < \infty, \quad i \in G_1.$$

Proof. It suffices to follow the proof of Theorem 1.2 by using (2.10) and Lemma 3.6. We omit the details. \square

Next, we show that, when the transition matrix P is reducible, $D_0(\mu)$ is dependent on the initial probability vector. For this, we need the following observation which is a consequence of the arguments in [7, Example 4.1].

Proposition 3.8. *Let $\nu_i, 1 \leq i \leq N$, be Borel probability measures on \mathbb{R}^q of compact support such that, for all $\epsilon > 0$, we have*

$$\max_{1 \leq i \leq N} \sup_{x \in \mathbb{R}^q} \nu_i(B(x, \epsilon)) \leq d_1 \epsilon^{d_2}.$$

Assume that $D_0(\nu_i) = t_i > 0, i \in G_1$. Let $(q_i)_{i=1}^N$ be a probability vector with $q_i > 0$ for all $i \in G_1$. Then for $\nu = \sum_{i=1}^N q_i \nu_i$, we have $D_0(\nu) = t_0$, where

$$t_0 = \frac{t_1 t_2 \cdots t_N}{q_1 t_2 \cdots t_N + \cdots + q_N t_1 \cdots t_{N-1}}.$$

Moreover, we have, $\underline{Q}_0^{t_0}(\nu) > 0$ if $\underline{Q}_0^{t_i}(\nu_i) > 0$ for all $1 \leq i \leq N$; and $\overline{Q}_0^{t_0}(\nu) < \infty$ if $\overline{Q}_0^{t_i}(\nu_i) < \infty$ for all $1 \leq i \leq N$.

Proof. We denote by $[x]$ the largest integer not greater than $x \in \mathbb{R}$. By the arguments in [7, Example 4.1], we have

$$(3.21) \quad \hat{e}_n(\nu) \begin{cases} \geq q_1 \hat{e}_n(\nu_1) + \cdots + q_N \hat{e}_n(\nu_N) \\ \leq q_1 \hat{e}_{[\frac{n}{N}]}(\nu_1) + \cdots + q_N \hat{e}_{[\frac{n}{N}]}(\nu_N) \end{cases}.$$

By (1.1) and (3.21), we easily see that $D_0(\nu) = t_0$. Furthermore, by (3.21),

$$(3.22) \quad Q_n(\nu, t_0) \geq t_0^{-1} \log n + \sum_{i=1}^N q_i \hat{e}_n(\nu_i) = \sum_{i=1}^N q_i Q_n(\nu_i, t_i).$$

Set $C_5 := (q_1 t_1^{-1} + \dots + q_N t_N^{-1}) \log(2N)$. Then, for all $n \geq 2N$, we have

$$(3.23) \quad Q_n(\nu, t_0) \leq t_0^{-1} \log n + \sum_{i=1}^N q_i \hat{e}_{[\frac{n}{N}]}(\nu_i) = \sum_{i=1}^N q_i Q_{[\frac{n}{N}]}(\nu_i, t_i) + C_5.$$

By (3.22) and (3.23), we conclude

$$\prod_{i=1}^N (\underline{Q}_0^{t_i}(\nu_i))^{q_i} \leq \underline{Q}_0^{t_0}(\nu) \leq \overline{Q}_0^{t_0}(\nu) \leq e^{C_5} \prod_{i=1}^N (\overline{Q}_0^{t_i}(\nu_i))^{q_i}.$$

This implies the second assertion of the proposition. \square

Example 3.9. Let $Q_1 = (q_{ij})_{i,j=1}^2, Q_2 = (t_{ij})_{i,j=3}^4$ be positive row-stochastic matrices ($q_{ij} > 0, 1 \leq i, j \leq 2$ and $t_{ij} > 0, 3 \leq i, j \leq 4$). Let P denote the block diagonal matrix $\text{diag}(Q_1, Q_2)$. Then P is reducible. Let $(c_{ij})_{4 \times 4}$ be given and assume that (1.3) holds. Let μ be the Markov-type measure associated with P and initial probability vector $(q_i)_{i=1}^4$. Clearly, Q_1, Q_2 are both irreducible. Let $(v_i^{(h)})_{i=1}^2$ be the normalized positive left eigenvector of Q_h for $h = 1, 2$. Write

$$t_1 := \frac{\sum_{i=1}^2 v_i^{(1)} \sum_{j=1}^2 q_{ij} \log q_{ij}}{\sum_{i=1}^2 v_i^{(1)} \sum_{j=1}^2 q_{ij} \log c_{ij}}; \quad t_2 := \frac{\sum_{i=3}^4 v_i^{(2)} \sum_{j=3}^4 t_{ij} \log t_{ij}}{\sum_{i=3}^4 v_i^{(2)} \sum_{j=3}^4 t_{ij} \log c_{ij}}.$$

By Proposition 3.7, we have, $0 < \underline{Q}_0^{t_1}(\mu_i) \leq \overline{Q}_0^{t_1}(\mu_i) < \infty$ for $i = 1, 2$; and for $i = 3, 4$, $0 < \underline{Q}_0^{t_2}(\mu_i) \leq \overline{Q}_0^{t_2}(\mu_i) < \infty$. Set

$$t_0 := \frac{t_1 t_2}{(q_1 + q_2)t_2 + (q_3 + q_4)t_1}.$$

Then by Proposition 3.8, we have, $0 < \underline{Q}_0^{t_0}(\mu) \leq \overline{Q}_0^{t_0}(\mu) < \infty$. In this example, $D_0(\mu)$ depends on the initial probability vector provided that $t_1 \neq t_2$.

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